The Local-to-global Theorem for Cocycle Expansion

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1 The local-to-global theorem for cocycle expanders

Constructing coboundary expanders is notoriously difficult, primarily because it is hard to design bounded-degree complexes that satisfy the condition $B^i = Z^i$. However, for many applications this requirement can be relaxed. Complexes that meet these weaker expansion conditions are known as cocycle expanders.

Definition 1.1. A simplicial complex X is a $(i+1)$-dimensional η-cocycle expander if

$$\min_{f \in C^i \setminus Z^i} \frac{|\delta_i f|}{|f - Z^i|} \geq \eta.$$

Analogous to how trickle-down theorem tells us that expansion of the links implies expansion of the global graph, it turns out that coboundary expansion on the links together with local-spectral expansion on the entire complex implies cocycle expansion. This is captured by the following theorem.

Theorem 1.2 ([KKL14, EK24, DD24])**.** *For any constants $\beta > 0$ and $\gamma > 0$ sufficiently small as a function of β, if X is a 3-dimensional simplicial complex with bounded vertex degree D [1] such that*

1. *X is a 3-dimensional γ-local spectral expander and*
2. *for all $v \in X(0)$, the link X_v is a 2-dimensional β-coboundary expander,*

then X is a 2-dimensional η-cocycle expander for $\eta = \min\left(\frac{1}{D}, \frac{\beta}{17}\right)$.

This result generalize to higher dimensional cocycle-expansion, but in this note we focus on 2-dimensional setting. The proof we present here combines ideas from [EK24, DD24].

1.1 The local correction algorithm

The proof of this theorem uses the local correction argument. The gist is that given any function $f \in C^1$, we iteratively make local corrections to f so that at each step the coboundary size $|\delta_1 f|$ decreases while the correction function has bounded weight. This procedure ends when we get a locally minimal function, defined as follows.

Definition 1.3. A function $f \in C^1$ is locally minimal if for every vertex $v \in X(0)$ and any localized function $h_v : X(1) \to \mathbb{F}_2$ such that $h_v(e) = 0,\ \forall e \not\ni v$ (in other words h_v can be nonzero only over v's adjacent edges), f satisfies that

$$|\delta_1 f| \leq |\delta_1(f + h_v)|.$$

The following observation of locally minimal function will be key in our proof.

[1] meaning every v is in at most D 3-faces.

Observation 1.4. *For any function $f \in C^1(X)$ and any vertex $v \in X(0)$, we define the induced function $f_v \in C^1(X_v)$ to be*

$$f_v(uw) = f(uw), \ \forall uw \in X_v(1)$$

and the projection function $g \in C^0(X_v)$ to be

$$g_v(u) = f(uv), \forall u \in X_v(0).$$

If f is locally minimal, then the local functions satisfy that

$$\left|f_v - B_v^1\right| = \left|f_v - \delta_0 g_v\right|,$$

where B_v^1 denote the 1-dimensional coboundary space of X_v.

In other words, $\delta_0 g_v$ is a closest function to f_v in B_v^1.

Now we formally define the algorithm that locally correct any function in C^1 to a locally minimal function.

The local correction algorithm Let $f \in C^1$ be the input. Set $f_0 = f$

At iteration i, if there exists $v_i \in X(0)$ and $h_i : X(1) \to \mathbb{F}_2$, which is only nonzero over adjacent edges of v_i, such that $|\delta_1(f_i + h_i)| < |\delta_1 f_i|$, update $f_{i+1} = f_i + h_i$. Repeat this until no such v_i, h_i exist.

First note that this process terminates since in every iteration the number of failed triangle test decreases. Let T be the number of iterations on input f, so the final output is f_T.

Two cases If the algorithm outputs a function $f_T \in Z^1$, then by the bound on the distance $\|f - f_T\|$ above we deduce that

$$\frac{|\delta_1 f|}{|f - Z^1|} \geq \frac{\sum_{i=1}^{T} |\delta_1 f_i| - |\delta_1 f_{i-1}|}{\sum_{i=1}^{T} \|h_i\|} \geq \min_i \frac{\min_{ab \in X_{v_i}(1)} \pi_2(v_i ab)}{\pi_1(\{e \in X(1) \mid e \ni v_i\})} \geq \frac{1}{D}.$$

Here the last equality holds for all bounded-degree simplicial complex with uniform distribution on their maximal faces. In fact, a more careful argument can remove this assumption and yield $\Theta(\beta)$ using coboundary expansion of X_v's. We omit it here and you will prove it in the exercise.

If $f_T \notin Z^1$, then we shall prove that in this case:

Claim 1.5. *For any simplicial complex X that satisifies the conditions in Theorem 1.2, any locally minimal function $f \in C^1 \setminus Z^1$ has $|\delta_1 f| > \frac{\beta}{17}$.*

Using this lower bound for the failure probability of locally minimal functions, we can derive that in this case

$$\frac{|\delta_1 f|}{|f - Z^1|} > \frac{\beta}{17}.$$

From the two case analysis we can conclude that X is a 2-dimensional η-cocycle expander, where $\eta = \min\left(\frac{1}{D}, \frac{\beta}{17}\right)$.

1.2 Proof of Claim 1.5

Given a locally minimal $f \in C^1 \setminus Z^1$, denote by $T^* = \{t \in X(2) \mid \delta_1 f(t) \neq 0\}$ the set of failed triangle tests of f. We shall prove that assume for the sake contradiction, $|\delta_1 f| = \left\|\vec{1}_{T*}\right\|^2 \leq \frac{\beta}{17}$ then there is an operator $P : \ell^2(X(2)) \to \ell^2(X(2))$ with $\|P\|_{op} \leq \gamma$ so that

$$\left(\frac{1}{12} - O\left(\frac{\gamma^2}{\beta^2}\right) - O(\gamma^2)\right) \cdot \left\|\vec{1}_{T^*}\right\|^2 \leq \left\langle \vec{1}_{T^*}, P\vec{1}_{T^*} \right\rangle. \tag{1}$$

Then by the assumption that γ is small we can derive the contradiction that

$$\left(\frac{1}{12} - O\left(\frac{\gamma^2}{\beta^2}\right) - O(\gamma^2)\right) \cdot \left\|\vec{1}_{T^*}\right\|^2 \leq \left\langle \vec{1}_{T^*}, P\vec{1}_{T^*} \right\rangle \leq \gamma \cdot \left\|\vec{1}_{T^*}\right\|^2.$$

To prove (1), we set $P = M_2^+ - U_1 D_2$, where M_2^+ is the non-lazy up-down walk over triangles and $U_1 D_2$ is the standard down-up walk on triangles. First note that by the γ-local spectral expansion of X, the norm of their difference is indeed $\left\|M_2^+ - U_1 D_2\right\|_{op} \leq \gamma$ (Lemma A.3).

Next, we notice that $\vec{1}_{T^*} = \delta_1 f$. So for any $\sigma \in X(3)$ we have $\delta_2 \vec{1}_{T^*}(\sigma) = 0$. In other words, if $t \in T^*$, than for any $\sigma \in X(3)$ that contains t, at least one of the other 3 triangles in σ must also be in T^*. Therefore we can derive

$$\left\langle \vec{1}_{T^*}, M_2^+ \vec{1}_{T^*} \right\rangle \geq \frac{1}{3} \cdot \left\|\vec{1}_{T^*}\right\|^2.$$

We next prove that

$$\left\langle \vec{1}_{T^*}, U_1 D_2 \vec{1}_{T^*} \right\rangle \leq \left(\frac{1}{4} + O\left(\frac{\gamma^2}{\beta^2}\right) + O(\gamma^2)\right) \cdot \left\|\vec{1}_{T^*}\right\|^2.$$

We rewrite the inner product as follows.

$$\begin{aligned}\left\langle \vec{1}_{T^*}, UD\vec{1}_{T^*} \right\rangle &= \mathbb{E}_{ab\sim\pi(1)}[\Pr_{c,d\sim\pi_{ab}(0)}[abc, abd \in T^*]] \\ &= \mathbb{E}_{a\sim\pi(0), b\sim\pi_a(0)}[\mathbb{E}_{t\supset ab}[\delta_1 f(t)]^2].\end{aligned}$$

We call an edge ab heavy if the local failure probability $\mathbb{E}_{t\supset ab}[\delta_1 f(t)]$ is large ($\geq \frac{1}{4}$). Similarly, we call a vertex a heavy if the local failure probability $\mathbb{E}_{t\ni a}[\delta_1 f(t)]$ is large($\geq \frac{1}{16}$).

To bound the inner product, we need to show that the edge failure probability concentrates well around the expectation ε, so there aren't many heavy edges. An edge ab could be heavy for one of the two reasons: 1. either a or b is a heavy vertex; 2. or despite a and b both not heavy, the edge ab does not "sample" the failure probability at the vertices well.

Specifically, denote the first bad event to be

$$E_1 = \left\{ab \in X(1) \mid \Pr_{t\ni a}[\delta_1 f(t) \neq 0] \geq \frac{1}{16} \text{ or } \Pr_{t\ni b}[\delta_1 f(t) \neq 0] \geq \frac{1}{16}\right\},$$

and the second bad event to be

$$E_2 = \left\{ab \in X(1) \mid ab \notin E_1 \text{ but } \Pr_{t\supset ab}[\delta_1 f(t) \neq 0] \geq \frac{1}{4}\right\}.$$

If ab is in neither E_1 or E_2, we have that $\Pr_{t\supset ab}[\delta_1 f(t) \neq 0] < \frac{1}{4}$.

So now we bound the inner product by

$$\begin{aligned}\left\langle \vec{1}_{T^*}, UD\vec{1}_{T^*} \right\rangle &\leq \Pr_{ab\sim\pi(1)}[E_1] + \Pr_{ab\sim\pi(1)}[E_2] \\ &\quad + \mathbb{E}_{ab\sim\pi(1)}\left[\mathbf{1}[\neg E_1 \wedge \neg E_2] \cdot \frac{1}{4}\mathbb{E}_{t\supset ab}[\delta_1 f(t)]\right] \\ &\leq \Pr_{ab\sim\pi(1)}[E_1] + \Pr_{ab\sim\pi(1)}[E_2] + \frac{1}{4}\left\|\vec{1}_{T^*}\right\|^2\end{aligned}$$

So we are left to bound the probability of E_1 and E_2 for locally minimal f.

Claim 1.6. *For an X that satisfies the conditions in Theorem 1.2, any locally minimal $f \in C^1 \setminus Z^1$ with $\left\|\vec{1}_{T^*}\right\|^2 := \varepsilon \leq \frac{\beta}{17}$ satisfies that*

$$\Pr_{ab\sim\pi(1)}[ab \in E_1] \leq \varepsilon \cdot O\left(\frac{\gamma^2}{\beta^2}\right) .$$

Proof. First define the set of heavy vertices to be

$$V^* = \left\{ v \in X(0) \mid \Pr_{t\ni v}[\delta_1 f(t) \neq 0] \geq \frac{1}{16} \right\}.$$

We define another related set

$$R = \left\{ v \in X(0) \mid \Pr_{t\in X_v(2)}[\delta_1 f(t) \neq 0] \geq \frac{\beta}{16} \right\}.$$

We shall first show that $R \supseteq V^*$ and then bound the weight of R.

For every $v \in V^*$ define the induced local functions $g_v : X_v(0) \to \mathbb{F}_2$ as

$$g_v(a) = f(\{a, v\}).$$

By construction in the link of v

$$\delta_0 g_v(\{a,b\}) \neq f(\{a,b\}) \Rightarrow f(\{v,a\}) + f(\{v,b\}) + f(\{a,b\}) \neq 0 \Rightarrow \delta_1 f(\{a,b,v\}) \neq 0.$$

Furthermore by local minimality of f, $\left|f_v - B_v^1\right| = |f_v - \delta_0 g_v| = \Pr_{t\ni v}[\delta_1 f(t) \neq 0]$. Since X_v is a 2-dimensional β-coboundary expander,

$$\frac{|\delta_1 f_v|}{|f_v - B_v^1|} \geq \beta \Rightarrow |\delta_1 f_v| \geq \beta \cdot \Pr_{t\ni v}[\delta_1 f(t) \neq 0].$$

So for every $v \in V^*$, we have that

$$\Pr_{t\in X_v(2)}[\delta_1 f(t) \neq 0] \geq \beta \cdot \Pr_{t\ni v}[\delta_1 f(t) \neq 0] \geq \frac{\beta}{16}.$$

Therefore $v \in R$ as well.

We next bound the probability that a random vertex is in R. Consider the bipartite graph G between $X(0)$ and $X(2)$ where we connect v and t if and only if $\{v\} \cup t \in X(3)$. Use σ_2 to denote the second largest singular value of the normalized biadjacency matrix of G. Then by the bipartite expander sampling lemma (Lemma A.1) we have that

$$\Pr_{v\sim\pi(0)}\left[\Pr_{t\in X_v(2)}[\delta_1 f(t) \neq 0] \geq \frac{\beta}{16}\right] \leq \varepsilon \cdot \frac{\sigma_2^2}{\left(\frac{\beta}{16} - \varepsilon\right)^2}$$

We thereby deduce that

$$\Pr_{v\sim\pi(0)}[v \in V^*] \leq \Pr_{v\sim\pi(0)}[v \in R] \leq \varepsilon \cdot \frac{\sigma_2^2}{(\beta/16 - \varepsilon)^2}.$$

By the spectral bound for the swap walks covered in Max's lectures, know that $\sigma_2^2 \leq 3\gamma^2$ (Lemma A.2). Thus we conclude by observing

$$\Pr_{ab\sim\pi(1)}[ab \in E_1] \leq 2 \Pr_{v\sim\pi(0)}[v \in R] \leq \varepsilon \cdot O\left(\frac{\gamma^2}{\beta^2}\right) .$$

□

The other claim needed in step 2 is the following.

Claim 1.7. *For an X that satisfies the conditions in Theorem 1.2, any locally minimal $f \in C^1 \setminus Z^1$ with $\left\|\vec{1}_{T^*}\right\|^2 = \varepsilon$ satisfies that*

$$\Pr_{ab\sim\pi(1)}[E_2] \leq \varepsilon \cdot O(\gamma^2).$$

Proof. For any edge $ab \in E_2$ we have that by the local minimality of f, the local failure probability at ab is $\Pr_{t\supset ab}[\delta_1 f(t)] \in [\frac{1}{4}, \frac{1}{2}]$. So in the link of ab, the set of triangles $T^* \cap \{t \in X(2) \mid t \supset ab\}$ induces a large cut C_{ab} in the link of ab. By the expander mixing lemma, the C_{ab} contains at least $\frac{1}{4} \cdot \frac{3}{4} - \gamma$-fraction of edges. For any $cd \in C_{ab}$, we know that exactly one of abc and abd are in T^*, so exactly one of acd and bcd must be in T^* as well. Wlog assume that for at least half of the $cd \in C_{ab}$, we have $acd \in T^*$, i.e.

$$\Pr_{cd\in X_{ab}(1)}[acd \in T^*] \geq \frac{3}{32} - \frac{\gamma}{2},$$

while

$$\Pr_{cd\in X_a(1)}[acd \in T^*] < \frac{1}{16}.$$

For such a fixed vertex a, to bound the probability $\Pr_{b\in X_a(0)}\left[\Pr_{cd\in X_{ab}(1)}[acd \in T^*] \geq \frac{3}{32} - \frac{\gamma}{2}\right]$, we consider the bipartite graph G_a between $X_a(0)$ and $X_a(1)$ where we connect $b \in X_a(0)$ and $cd \in X_a(1)$ if and only if $bcd \in X_a(2)$. Use σ_a to denote the second largest singular value of the normalized biadjacency matrix of G_a. Then by the bipartite graph expander sampling lemma we get that

$$\Pr_{b\in X_a(0)}\left[\Pr_{cd\in X_{ab}(1)}[acd \in T^*] \geq \frac{3}{32} - \frac{\gamma}{2}\right] \leq \Pr_{cd\in X_a(1)}[acd \in T^*] \cdot \frac{\sigma_a^2}{\left(\frac{1}{32} - \frac{\gamma}{2}\right)^2}.$$

Averaging over all non-heavy vertices a we obtain

$$\begin{aligned}
\Pr_{ab\sim\pi(1)}[E_2] &\leq \mathbb{E}_{a\sim\pi(0)}\left[\Pr_{b\in X_a(1)}[ab \text{ is heavy}] \cdot \vec{1}[a, b \text{ not heavy}] \cdot \vec{1}\left[\Pr_{cd\in X_{ab}(1)}[acd \in T^*] \geq \frac{3}{32} - \frac{\gamma}{2}\right]\right] \\
&\quad + \mathbb{E}_{a\sim\pi(0)}\left[\Pr_{b\in X_a(1)}[ab \text{ is heavy}] \cdot \vec{1}[a, b \text{ not heavy}] \cdot \vec{1}\left[\Pr_{cd\in X_{ab}(1)}[bcd \in T^*] \geq \frac{3}{32} - \frac{\gamma}{2}\right]\right] \\
&\leq 2\mathbb{E}_{a\sim\pi(0)}\left[\Pr_{b\in X_a(1)}\left[\vec{1}[a \text{ not heavy}] \cdot \Pr_{cd\in X_{ab}(1)}[acd \in T^*] \geq \frac{3}{32} - \frac{\gamma}{2}\right]\right] \\
&\leq 2\mathbb{E}_{a\sim\pi(0)}\left[\Pr_{cd\in X_a(1)}[acd \in T^*]\right] \cdot \frac{\max_{a\in X(0)} \sigma_a^2}{\left(\frac{1}{32} - \frac{\gamma}{2}\right)^2} \\
&= \varepsilon \cdot O(\gamma^2)
\end{aligned}$$

The last inequality follows from the the spectral expansion of swap walks in γ-local spectral expanders (Lemma A.2). □

A Auxiliary lemmas

In the proof we used the following results. We omit their proofs because they are either standard or have been covered in an earlier talk.

Lemma A.1. *Let $G = (L, R, E)$ be a biregular bipartite graph whose normalized biadjacency operator has second singular value at most σ. For every $f : L \to \mathbb{R}$, let*

$$\mu = \mathbb{E}_{v\sim L}[f(v)].$$

Then, for every $\varepsilon > 0$,

$$\Pr_{u\sim R}\left[\mathbb{E}_{v\sim N(u)}[f(v)] > \mu + \varepsilon\right] \leq \frac{\sigma^2 \operatorname{Var}_{v\sim L}[f(v)]}{\varepsilon^2}.$$

Lemma A.2 ([DD19])**.** *Let X be a d-dimensional (two-sided) λ-local spectral expander. Let $i, j \geq 0$ satisfy $i + j + 1 \leq d$, and let*

$$S_{i,j} : L^2(X(i), \pi_i) \longrightarrow L^2(X(j), \pi_j)$$

be the normalized (i, j)-swap walk operator. Then

$$\sigma_2(S_{i,j}) \leq (i + 1)(j + 1)\lambda.$$

Lemma A.3 ([DDFH24])**.** *Let X be a d-dimensional (two-sided) λ-local spectral expander. Then, for every $0 \leq i \leq d - 1$,*

$$\left\|M_i^+ - U_{i-1}D_i\right\|_{op} \leq \lambda,$$

where M_i^+ is the non-lazy up-down walk on $X(i)$, and the operator norm is with respect to $L^2(X(i), \pi_i)$.

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