Coboundary and Cocycle Expanders

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Local-spectral expander review

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A 2-dimensional simplicial complex X is a λ-local spectral expander if every vertex link is a λ-spectral expander and the 1-skeleton of X is a λ-spectral expander.

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Today, we consider a different notion of high-dimensional expansion.

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i-cochain space

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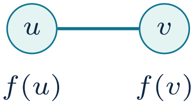
i-th coboundary operator

$$\delta_i : C^i \to C^{i+1}, \qquad (\delta_i g)(t) = \sum_{e \subset t} g(e).$$

δ_0: $C^0 \to C^1$



$$(\delta_0 f)(uv) = f(u) + f(v)$$

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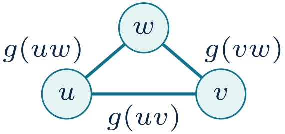

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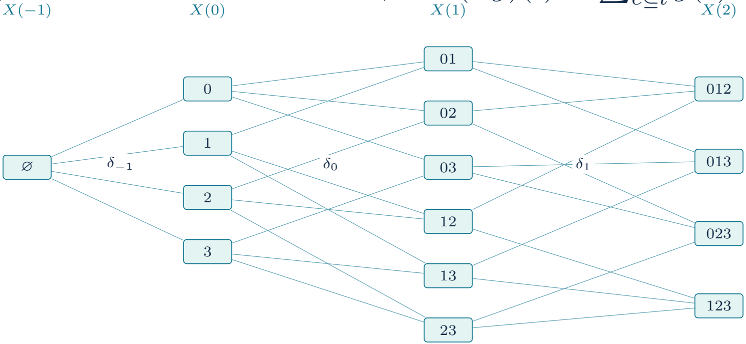
Function spaces over X: Hasse diagram

Coboundary operators $\delta_i : C^i \to C^{i+1}, \quad (\delta_i g)(t) = \sum_{e \subseteq t} g(e).$



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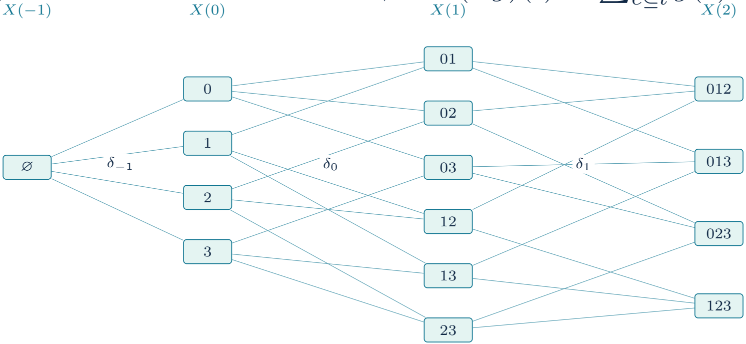

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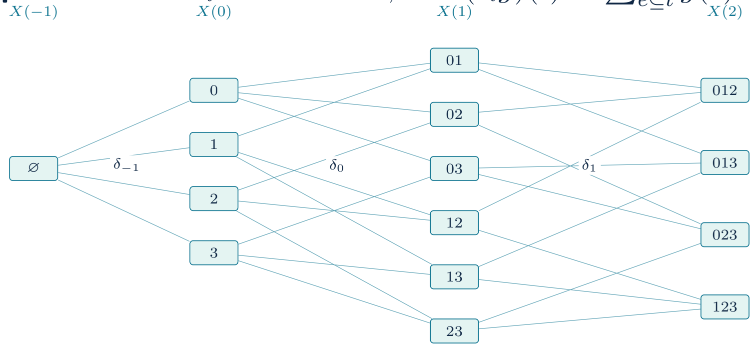

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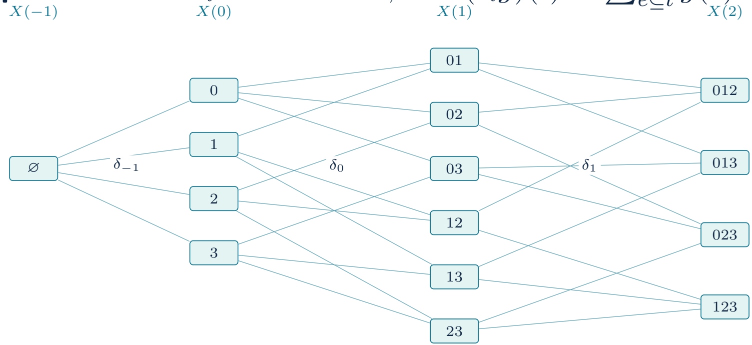

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This is called the i-th boundary operator.

Composition of coboundary maps

$$(\delta_i g)(t) = \sum_{e \subseteq t} g(e).$$

Quick exercise. For every $f \in C^0$,

$$\delta_1 \delta_0 f = \mathbf{0}.$$

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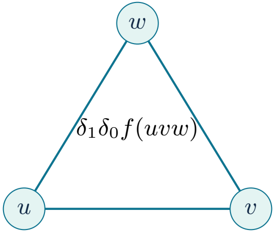

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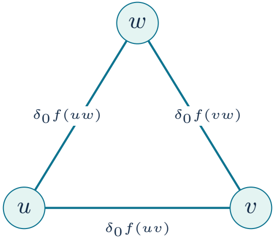

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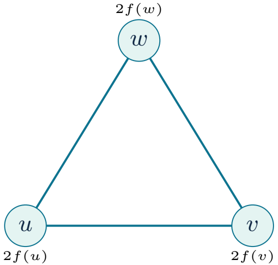

$$(\delta_1 \delta_0 f)(uvw) = 2f(u) + 2f(v) + 2f(w)$$

Coboundary space and cocycle space

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$$i\text{-th coboundary space } B^i := \mathrm{Im}(\delta_{i-1}) \subseteq C^i,$$

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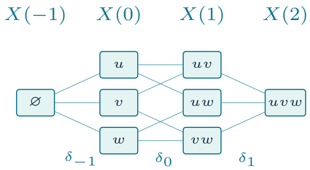

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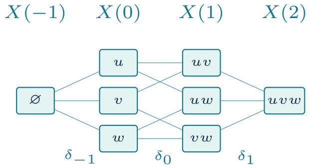

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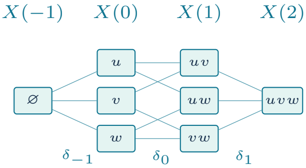

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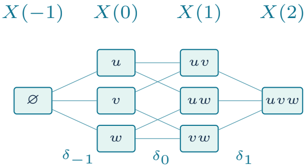

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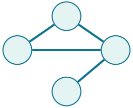

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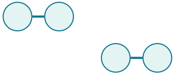

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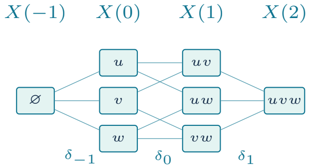

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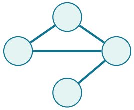

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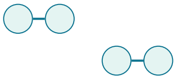

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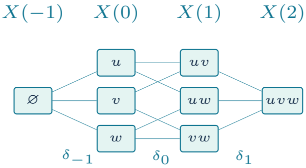

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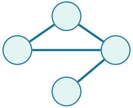

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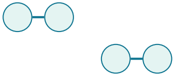

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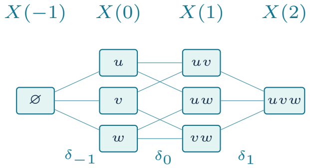

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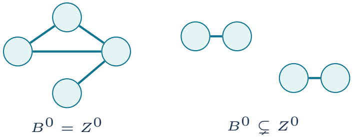

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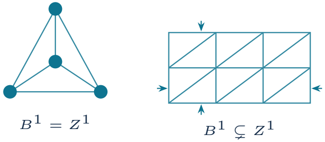

Coboundary expanders

Let $|f|$ denote the fraction of faces on which f is nonzero, and let $|f - B^i|$ denote the distance from f to the closest function in B^i.

Coboundary expander [LM06, Gro10]

X is an $(i+1)$-dimensional η-coboundary expander if $$\min_{f \in C^i \setminus B^i} \frac{|\delta_i f|}{|f - B^i|} \geq \eta.$$

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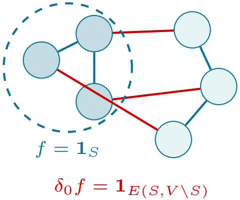

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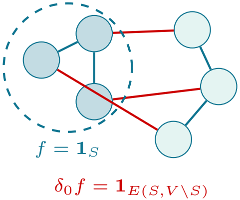

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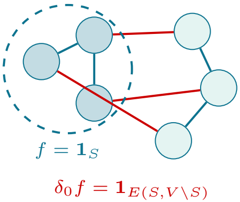

A graph has conductance c if and only if it is a 1-dimensional $2c$-coboundary expander.

2-dimensional coboundary expanders

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Example: complete complexes. A 2-dimensional complete complex is a 2-dimensional 1-coboundary expander.

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$$B^1 = \{ \mathbf{1}_{E(S, V \setminus S)} : S \subseteq V \},$$

  the set of complete bipartite graphs.

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- Create one variable x_v for each vertex $v \in V$.
- For each edge $uv \in E$, pick $f(uv) \in \mathbb{F}_2$ and add the equation

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More generally, an $(i+1)$-dimensional coboundary expander gives the subspace B^i a local membership test whose rejection probability is proportional to the distance $|f - B^i|$.

Cocycle expanders

We can also define a natural notion of expansion with respect to the cocycle space Z^i.

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$$|E_j(S, V_j \setminus S)| = |\delta_0 f| \cdot |E|, \text{ and } \min(d(S), d(V_j \setminus S)) = |f - Z^0| \cdot 2|E|$$

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Each connected component of a graph has conductance $\geq c$ if and only if the graph is a 1-dimensional $2c$-cocycle expander.

Constructions of cocycle expanders

- LSV Ramanujan complexes give bounded-degree cocycle expanders in every dimension [KKL14, EK24, DD24].
- Kaufman-Oppenheim complexes give bounded-degree 2-dimensional cocycle expanders [KO21].
- B_3-type Chevalley coset complexes give bounded-degree 2-dimensional cocycle expanders [OS24].
- Kac-Moody-Steinberg complexes give bounded-degree cocycle expanders in every dimension [OVB25a].

Constructions of coboundary expanders

- **Simple constructions:** complete complexes and complete multipartite complexes.
- **Random constructions:** dense random complexes [DK12, LLR19].
- **Spherical buildings** [Gro10, LMM16, DD24].
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Open problem

Construct bounded-degree coboundary expanders in every dimension.

Applications of coboundary and cocycle expansion

Generalization of coboundary and cocycle expansion underlies applications to:

- **Locally testable codes** with constant-rate, constant-relative distance, constant-locality [DEL$^+$22].
- **Quantum LDPC codes** with constant rate and constant distance [PK22].
- **Agreement tests in low acceptance regime** [BLM24, DDL24].

The cone method for proving coboundary expansion

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Idea. Given $f \in C^1$, find $g \in C^0$ such that

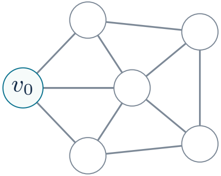
- $\delta_0 g$ is close to f,
- if $f(uv) \neq \delta_0 g(uv)$, $\exists\, t \in X(2)$, s.t. $\delta_1 f(t) = 1$.

Thus connect $|f - \delta_0 g|$ to $|\delta_1 f|$.

The cone method

A cone on X [Gro10, LMM16, KM19, KO21]

1. A base vertex v_0.



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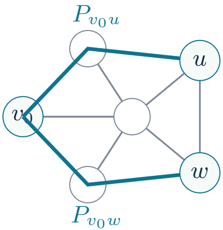

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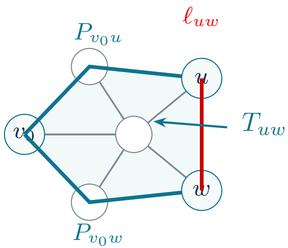
3. For each $uw \in X(1)$, a set of triangles $T_{uw} \subset X(2)$ filling the loop $\ell_{uw} = P_{v_0 u} \circ uw \circ P_{v_0 w}^{-1}$.

$$\delta_1^* \mathbf{1}_{T_{uw}} = \mathbf{1}_{\ell_{uw}}, \quad \ell_{uw} \text{ is the boundary of } T_{uw}.$$



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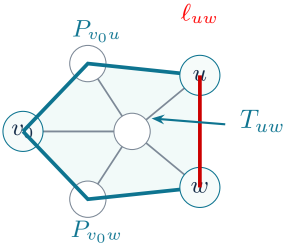
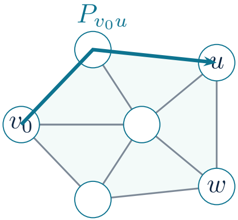
Such fillings exist whenever $B^1 = Z^1$.



Define g from a cone

Given a cone $C = \left(v_0, \{P_{v_0 u}\}_{u \neq v_0}, \{T_{uw}\}_{uw \in X(1)}\right)$, define $g \in C^0$ to be

$$g(v_0) = 0, \qquad g(u) = \sum_{e \in P_{v_0 u}} f(e).$$



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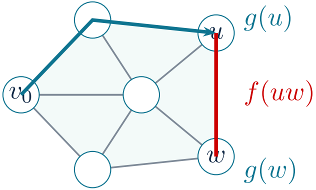

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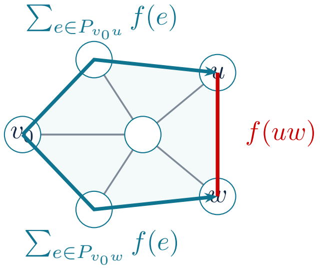

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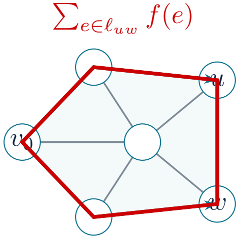

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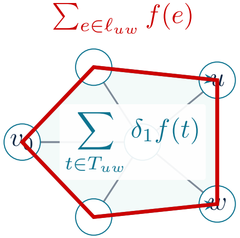

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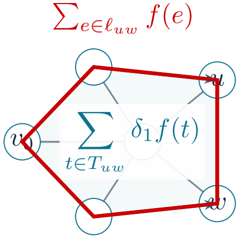

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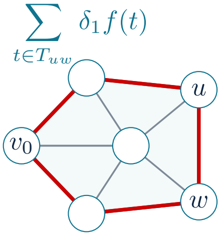

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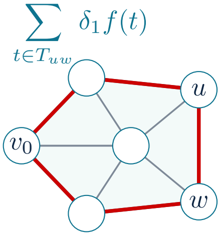

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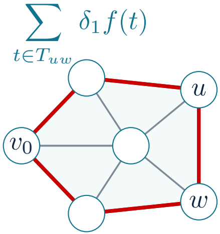

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Therefore, there is a triangle $t \in T_{uw}$ such that $\delta_1 f(t) = 1$.

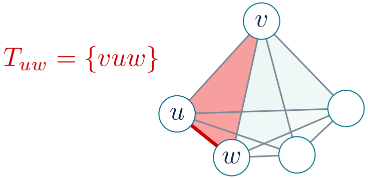
Example: the complete complex

In a complete complex, fix a base vertex v. Construct the cone

$$C_v = (v, \{P_{vu} = \{vu\}\}, \{T_{uw} = \{vuw\}\}) \, .$$

Define

$$g_v(v) = 0, \qquad g_v(u) = f(vu).$$



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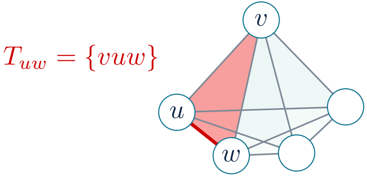

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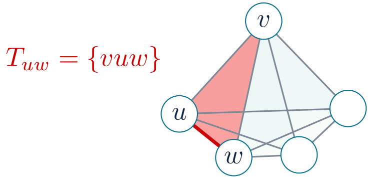

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Limitation: for fixed v, the cone C_v only uses triangles containing v. These are only an $\Theta(1/n)$ fraction of all triangles. We need to take into account other triangles.

Example: averaging over cones

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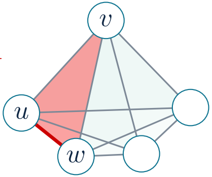

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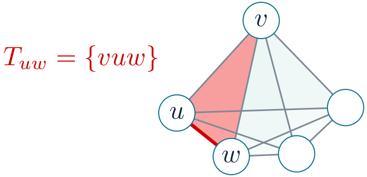

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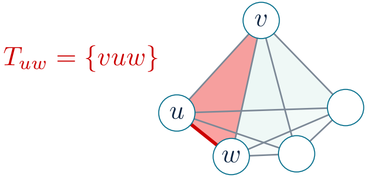

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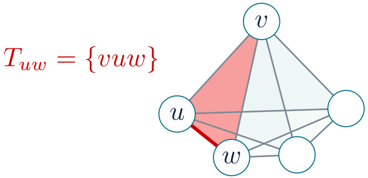

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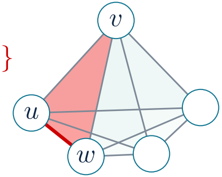

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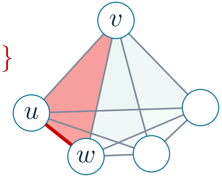

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$$\frac{|\delta_1 f|}{|f - B^1|} \geq \frac{|\delta_1 f|}{\mathbb{E}_v |f - \delta_0 g_v|} = \frac{|\delta_1 f|}{\mathbb{E}_v \Pr_{uw}[f(uw) \neq \delta_0 g_v(uw)]} \geq 1.$$

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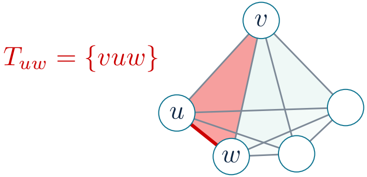

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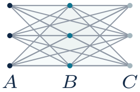
\end{aligned}$$

$$\frac{|\delta_1 f|}{|f - B^1|} \geq \frac{|\delta_1 f|}{\mathbb{E}_v |f - \delta_0 g_v|} = \frac{|\delta_1 f|}{\mathbb{E}_v \Pr_{uw}[f(uw) \neq \delta_0 g_v(uw)]} \geq 1.$$

Thus, the complete complex is a 2-dimensional 1-coboundary expander.

More coboundary expanders via the cone method

Complete multipartite complexes.
One can construct fillings with $|T_{uw}| \leq 4$, so the cone method gives $1/4$-coboundary expansion.



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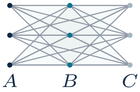

Spherical buildings.
A more involved cone method proves coboundary expansion for spherical buildings [LMM16, DD24].

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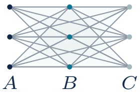

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To obtain constant coboundary expansion from the cone method, the complex needs to be dense and to have symmetry on faces.

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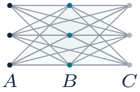

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How can one prove coboundary expansion for bounded-degree complexes? How can one prove cocycle expansion?

The answer: the local-to-global theorem for cocycle expansion

Theorem [EK24, KM21, DD24]

For every $\eta > 0$, and $\gamma > 0$ sufficiently small as a function of η, let X be a 3-dimensional simplicial complex such that:

1. X is a γ-local spectral expander;
2. every vertex link is a 2-dimensional η-coboundary expander.

Then X is a 2-dimensional $\Theta(\eta)$-*cocycle expander*.

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If, in addition, $B^1 = Z^1$, then cocycle expansion becomes coboundary expansion.

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